

# Analysis of the semileptonic $B_c \rightarrow B_u^* \ell^+ \ell^-$ decay from QCD sum rules

T.M. Aliev<sup>a</sup>, M. Savci<sup>b</sup>

Physics Department, Middle East Technical University, 06531 Ankara, Turkey

Received: 1 April 2006 /

Published online: 7 June 2006 – © Springer-Verlag / Società Italiana di Fisica 2006

**Abstract.** We analyze the semileptonic  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay in the framework of the Standard Model. We calculate the  $B_c$  to  $B_u^*$  transition form factors in QCD sum rules. Analytical expressions for the spectral densities and gluon condensates are presented. The branching ratio of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay is calculated, and it is obtained that this decay can be detectable at forthcoming LHC machines.

**PACS.** 12.60.-i; 13.30.-a.; 13.88.+e

## 1 Introduction

The double heavy  $B_c$  mesons were discovered by the CDF Collaboration [1], and were found to have a mass  $m_{B_c}^{\text{exp}} = (6.4 \pm 0.39 \pm 0.13)$  GeV. Recently the CDF Collaboration [2] announced an accurate determination of the  $B_c$  meson mass,  $m_{B_c} = (6.2857 \pm 0.0053 \pm 0.0012)$  GeV. The study of the  $B_c$  meson has received great interest, due to its special properties: firstly, its lowest bound state is composed of two heavy (charm and beauty) quarks with open flavor; secondly, this meson attracts the interest of physicists for checking predictions of perturbative QCD in the laboratory; and lastly, the weak decay channels of  $B_c$  meson are compared to the corresponding decay channels of  $B_q$  ( $q = u, d, s$ ) and can be divided into three classes:

- the  $b$  quark decaying weakly, with the  $c$  quark as spectator, e.g.,  $B_c \rightarrow J/\psi \ell \bar{\nu}_\ell$ ;
- the  $c$  quark decaying weakly, with the  $b$  quark as spectator, e.g.,  $B_c \rightarrow B_s \ell \bar{\nu}_\ell$ ;
- the annihilation channels like  $B_c \rightarrow \ell \bar{\nu}_\ell$ ,  $B_c \rightarrow D_s^{*-} \bar{K}^{0*}$ .

There are quite a few theoretical works studying the various leptonic, semileptonic and hadronic exclusive decay channels of  $B_c$  meson (for a review, see [3]).

From an experimental point of view, the study of weak semileptonic decays of the  $B_c$  meson is quite important for the determination of the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements, the leptonic decay constant  $f_{B_c}$ , etc.

Much more  $B_c$  mesons and more detailed information on their decay properties are expected at the forthcom-

ing LHC accelerator. In particular, this holds true for the dedicated detectors BTeV and LHCb which are specially designed for the analysis of  $B$  physics where one expects to see up to  $10^{10} B_c$  mesons per year with a luminosity of  $10^{34} \text{ cm}^{-2} \text{ s}^{-1}$  [4].

The rare decays constitute one of the most important classes of decays suitable for a more precise determination of the parameters of the SM, as well as for looking for new physics beyond the SM [5–7], since the flavor changing neutral currents (FCNC) are absent in the SM at tree level and appear only at loop level, due to the running of a virtual particle in the loop.

Of the FCNC processes involving  $K$  and  $B$  mesons, the main attention has been focused on  $B^0-\bar{B}^0$  mixing,  $b \rightarrow s \ell^+ \ell^-$ ,  $b \rightarrow s \gamma$ ,  $s \rightarrow d \ell^+ \ell^-$ ,  $s \rightarrow d \nu \bar{\nu}$ , etc., since a top quark runs in the loop.

The study of the FCNC decays in the charm sector has not received enough attention. This can be explained by the fact that in the SM,  $D^0-\bar{D}^0$  mixing [8, 9], as well as FCNC decays [10–12], are expected to be very small. Moreover, long distance effects are quite huge, since the loop in charm decay involves light down quarks.

In the present work we present a detailed analysis of the semileptonic  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay in the framework of the sum rules method. This mode is likely to be observed in forthcoming accelerator experiments. Note that the radiative decay  $B_c \rightarrow B_u^* \gamma$ , which at the quark level is described by the  $c \rightarrow u + \gamma$  transition, is investigated in [13, 14].

The plan of this work is as follows: In the following section QCD sum rules of the three-point correlators are considered and sum rules for the form factors that are responsible for the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay are constructed. In Sect. 3 we present our numerical results and conclusions.

<sup>a</sup> e-mail: taliev@metu.edu.tr

Permanent address: Institute of Physics, Baku, Azerbaijan

<sup>b</sup> e-mail: savci@metu.edu.tr

## 2 Sum rules for the transition form factors

In the present section we derive sum rules for the form factors that control the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay. This decay is described by the  $c \rightarrow u \ell^+ \ell^-$  transition at quark level. The matrix element for the  $c \rightarrow u \ell^+ \ell^-$  decay can be written in the following form:

$$\begin{aligned} \mathcal{M} = & \frac{G\alpha}{4\sqrt{2}\pi} \left[ C_9^{\text{eff}}(m_c) \bar{u} \gamma_\mu (1 - \gamma_5) c \bar{\ell} \gamma_\mu \ell \right. \\ & + C_{10}(m_c) \bar{u} \gamma_\mu (1 - \gamma_5) c \bar{\ell} \gamma_\mu \gamma_5 \ell \\ & \left. - 2m_c C_7(m_c) \bar{u} i \sigma_{\mu\nu} \frac{q^\nu}{q^2} (1 + \gamma_5) c \bar{\ell} \gamma_\mu \ell \right]. \end{aligned} \quad (1)$$

At  $\mu = m_c$ ,  $C_9^{\text{eff}}$  is given by [15]

$$\begin{aligned} C_9^{\text{eff}}(m_c) = & C_9(m_W) \\ & + \sum_{i=d,s} \lambda_i \left[ -\frac{2}{9} \ln \frac{m_i^2}{m_W^2} + \frac{8}{9} \frac{z_i^2}{\bar{s}} \right. \\ & \left. - \frac{1}{9} \left( 2 + \frac{4z_i^2}{\bar{s}} \right) \sqrt{\left| 1 - \frac{4z_i^2}{\bar{s}} \right|} T(z_i) \right], \end{aligned} \quad (2)$$

where

$$T(z_i) = \begin{cases} 2 \arctan \left[ \frac{1}{\sqrt{4z_i^2/\bar{s} - 1}} \right] & (\text{for } \bar{s} < 4z_i^2), \\ \ln \left| \frac{1 + \sqrt{1 - 4z_i^2/\bar{s}}}{1 - \sqrt{1 - 4z_i^2/\bar{s}}} \right| - i\pi & (\text{for } \bar{s} > 4z_i^2). \end{cases}$$

where  $\bar{s} = s/m_c^2$ ,  $z_i = m_i/m_c$ . The logarithmic dependence on the internal quark mass  $m_i$  in  $T(z_i)$  in (2) cancels a similar term in the function  $F_1(x_i)$  entering in  $C_9(m_W)$ , leaving no spurious divergences in the limit  $m_i \rightarrow 0$ . It should be noted here that QCD corrections do not effect  $C_{10}$ , i.e.,  $C_{10}(m_c) = C_{10}(m_W)$ .

The QCD corrections are particularly important for the Wilson coefficient  $C_7$  and in the numerical analysis we use the two-loop QCD-corrected value of  $C_7^{\text{eff}}$  which is calculated in [11].

The values of the Wilson coefficients at  $\mu = m_W$  are given by the following expressions (see for example [10, 13]):

$$\begin{aligned} C_7(m_W) = & - \sum_{i=d,s,b} \lambda_i F_2(x_i), \\ C_9(m_W) = & \frac{1}{s_W^2} \sum_{i=d,s,b} \lambda_i \left[ (C^{\text{box}}(x_i) + C^Z(x_i)) \right. \\ & \left. - 2s_W^2 (F_1(x_i) + C^Z(x_i)) \right], \\ C_{10}(m_W) = & - \frac{1}{s_W^2} \sum_{i=d,s,b} \lambda_i (C^{\text{box}}(x_i) + C^Z(x_i)), \end{aligned} \quad (3)$$

where  $\lambda_i = V_{ci} V_{ui}^*$ ,  $x_i = m_i^2/m_W^2$ . The functions  $F_1(x)$ ,  $F_2(x)$ ,  $C^{\text{box}}(x_i)$  and  $C^Z(x_i)$  are those derived in [16] and are all given in Appendix A.

Similar to the  $b \rightarrow s \ell^+ \ell^-$  transition, the Wilson coefficient in the  $c \rightarrow u \ell^+ \ell^-$  transition receives long dis-

tance contributions which have their origin in the real  $\bar{q}q$  intermediate state, i.e.,  $\rho$  and  $\omega$  mesons. These contributions can be written via the following replacement in  $C_9^{\text{eff}}(m_c)$  [17]:

$$C_9^{\text{eff}} \rightarrow C_9^{\text{eff}} + \frac{3\pi}{\alpha^2} \sum_i \kappa_i \frac{m_{V_i} \Gamma_{V_i \rightarrow \ell^+ \ell^-}}{m_{V_i}^2 - \hat{s} - im_{V_i} \Gamma_{V_i}},$$

where  $m_{V_i}$  and  $\Gamma_{V_i}$  are the resonance mass and width. The Fudge factor  $\kappa_i$  is determined in [10] to have the values  $\kappa_\rho = 0.7$  and  $\kappa_\omega = 3.1$ .

Having the matrix element for  $c \rightarrow u \ell^+ \ell^-$  transition at hand, our next problem is the calculation of the matrix element for the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay. It follows from (1) that, in order to calculate the amplitude of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay, the following matrix elements are needed:

$$\begin{aligned} & \langle B_u^* | \bar{u} \gamma_\mu (1 - \gamma_5) c | B_c \rangle, \\ & \langle B_u^* | \bar{u} i \sigma_{\mu\nu} q^\nu (1 + \gamma_5) b | B_c \rangle. \end{aligned}$$

These hadronic matrix elements of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay can be parametrized in terms of the form factors in the following way:

$$\begin{aligned} & \langle B_u^*(p', \varepsilon) | \bar{u} \gamma_\mu (1 - \gamma_5) c | B_c(p) \rangle \\ & = \epsilon_{\mu\nu\alpha\beta} \varepsilon^{*\nu} p^\alpha p'^\beta \frac{2V(q^2)}{m_{B_c} + m_{B_u^*}} \\ & - i \left[ \varepsilon_\mu^* (m_{B_c} + m_{B_u^*}) A_1(q^2) - (\varepsilon^* q) \mathcal{P}_\mu \frac{A_2(q^2)}{m_{B_c} + m_{B_u^*}} \right. \\ & \left. - (\varepsilon^* q) \frac{2m_{B_u^*}}{q^2} [A_3(q^2) - A_0(q^2)] q_\mu \right], \end{aligned} \quad (4)$$

$$\begin{aligned} & \langle B_u^*(p', \varepsilon) | \bar{u} \sigma_{\mu\nu} q^\nu (1 + \gamma_5) c | B_c(p) \rangle \\ & = 2i \epsilon_{\mu\rho\alpha\beta} \varepsilon^{*\rho} p^\alpha p'^\beta T_1(q^2) \\ & + \left[ \varepsilon_\mu^* (m_{B_c}^2 - m_{B_u^*}^2) - (\varepsilon^* q) \mathcal{P}_\mu \right] \times T_2(q^2) \\ & + (\varepsilon^* q) \left[ q_\mu - \frac{q^2}{m_{B_c}^2 - m_{B_u^*}^2} \mathcal{P}_\mu \right] T_3(q^2), \end{aligned} \quad (5)$$

where  $\mathcal{P}_\mu = (p + p')_\mu$ ,  $q_\mu = (p - p')_\mu$  and  $\varepsilon^*$  is the polarization 4-vector of the vector  $B_u^*$  meson. Note that the form factor  $A_3(q^2)$  can be written as a linear combination of  $A_1$  and  $A_2$ :

$$A_3(q^2) = \frac{m_{B_c} + m_{B_u^*}}{2m_{B_u^*}} A_1(q^2) - \frac{m_{B_c} - m_{B_u^*}}{2m_{B_u^*}} A_2(q^2), \quad (6)$$

and in order to guarantee the finiteness of the results at  $q^2 = 0$ ,  $A_3(0) = A_0(0)$  should be satisfied.

The identity

$$\sigma_{\mu\nu} \gamma_5 = -\frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta}$$

leads to the following relation between the form factors  $T_1$  and  $T_2$ :

$$T_1(0) = T_2(0).$$

Moreover, it is shown in [18] that the form factors  $T_1(q^2)$  and  $A_0(q^2)$  are also related, i.e.,

$$A_0(q^2) = -T_1(q^2),$$

and so are  $T_3(q^2)$ ,  $A_1(q^2)$  and  $A_2(q^2)$  through the relation

$$T_3(q^2) = - \left[ \frac{m_{B_c} + m_{B_u^*}}{2m_{B_u^*}} A_1(q^2) - \frac{m_{B_c}^2 + 3m_{B_u^*}^2 - q^2}{2m_{B_u^*}(m_{B_c} + m_{B_u^*})} A_2(q^2) \right]. \quad (7)$$

It follows from these relations that in describing the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay, we need to know only  $V$ ,  $A_1$  and  $A_2$ . In the further analysis the values of these form factors at  $q^2 = 0$  are needed (see Sect. 3). Note that at this point the form factor  $T_1(0)$  is calculated in [14], and hence we do not present its expression in this work.

As has already been noted, in order to calculate these form factors, appearing in the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay, we explore the three-point QCD sum rules [19].

For the evolution of the above-mentioned form factors in the framework of the QCD sum rules, we start with the following three-point correlation functions:

$$\Pi_{\nu\mu} = - \int d^4x d^4y e^{-i(p_1x - p_2y)} \times \langle 0 | \mathcal{T} \{ J_{B_u^* \nu}(y) J_\mu(0) J_{B_c}(x) \} | 0 \rangle, \quad (8)$$

where  $J_{B_u^*}^\nu = \bar{b}(y) \gamma^\nu u(y)$  is the interpolating current of the  $B_u^*$  meson,  $J_\mu = \bar{u}(0) \gamma_\mu (1 + \gamma_5) c(0)$ , and  $J_{B_c} = \bar{c}(x) i(1 + \gamma_5) b(x)$  is the interpolating current of the  $B_c$  meson. The contribution of the scalar part of this current to  $B_c$  is zero.

The Lorentz structures of these correlators can be written in the following form:

$$\Pi_{\nu\mu}^{(V+A)} = \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta \Pi_V + \Pi_{A_1} g_{\mu\nu} + \Pi_2 \mathcal{P}_\mu \mathcal{P}_\nu + \Pi_{A_2} \mathcal{P}_\nu q_\mu + \Pi_3 \mathcal{P}_\mu q_\nu + \Pi_4 q_\mu q_\nu. \quad (9)$$

The phenomenological part of these correlators can be calculated by inserting a complete set of intermediate states with the same quantum numbers as the currents  $J_{B_u^*}$  and  $J_{B_c}$  possess in the correlation functions (4) and (5), and expressing these functions as the sum of the contributions of the lowest lying and excited states, we get

$$\Pi_{\nu\mu}(p_1^2, p_2^2, q^2) = - \frac{\langle 0 | J_{B_u^* \nu} | B_u^* \rangle \langle B_u^* | J_\mu | B_c \rangle \langle B_c | J_{B_c} | 0 \rangle}{(p_1^2 - m_{B_c}^2)(p_2^2 - m_{B_u^*}^2)} + \text{contributions from higher states}. \quad (10)$$

The matrix elements in (10) are defined as follows:

$$\langle 0 | J_{B_u^*}^\mu | B_u^* \rangle = f_{B_u^*} m_{B_u^*} \varepsilon^{*\mu} \\ i \langle B_c | \bar{c}(1 + \gamma_5) b | 0 \rangle = \frac{f_{B_c} m_{B_c}^2}{m_b + m_c}. \quad (11)$$

Performing summation over the polarization of  $B_u^*$  meson on the matrix elements  $\langle B_u^* | \bar{u} \gamma_\mu (1 + \gamma_5) c | B_c \rangle$  that are given in (4), we get for the physical part

$$\Pi_{\nu\mu}^{(V+A)} = - \frac{f_{B_c} f_{B_u^*} m_{B_c}^2}{(m_b + m_c)(p_1^2 - m_{B_c}^2)(p_2^2 - m_{B_u^*}^2)} \times \left\{ \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta \frac{2V}{m_{B_c} + m_{B_u^*}} + i(m_{B_c} + m_{B_u^*}) \left( -g_{\mu\nu} + \frac{(\mathcal{P} - q)_\mu (\mathcal{P} - q)_\nu}{4m_{B_u^*}^2} \right) A_1 - \frac{i}{m_{B_c} + m_{B_u^*}} \mathcal{P}_\mu \left( -q_\nu + \frac{p' q (\mathcal{P} - q)_\nu}{2m_{B_u^*}^2} \right) A_2 - \frac{2im_{B_u^*}}{q^2} q_\mu \left( -q_\nu + \frac{p' q (\mathcal{P} - q)_\nu}{2m_{B_u^*}^2} \right) (A_3 - A_0) \right\}. \quad (12)$$

The expressions of the form factors  $V$ ,  $A_1$  and  $A_2$  can be determined by choosing the coefficients of the Lorentz structures  $\epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta$ ,  $ig_{\mu\nu}$  and  $\mathcal{P}_\mu q_\nu$  in the correlator function  $\Pi_{\nu\mu}^{(V+A)}$ , respectively.

On the other side, the three-point correlator function can be calculated by the operator product expansion (OPE) in the deep Euclidean region  $p_1^2 \ll (m_b + m_c)^2$ ,  $p_2^2 \ll m_b^2$ .

The time ordered products of the currents in the three-point correlator function in (1) can be expanded in terms of a series of local operators with increasing dimension, as is shown in the following:

$$- \int d^4x d^4y e^{-i(px - p'y)} \mathcal{T} \{ J_{B_u^* \nu} J_\mu J_{B_c} \} = (C_0)_{\nu\mu} I + (C_3)_{\nu\mu} \bar{q}q + (C_4)_{\nu\mu} G_{\alpha\beta} G^{\alpha\beta} + (C_5)_{\nu\mu} \bar{q} \sigma_{\alpha\beta} G^{\alpha\beta} q + (C_6)_{\nu\mu} \bar{q} \Gamma q \bar{q} \Gamma' q, \quad (13)$$

where  $(C_i)_{\nu\mu}$  are the Wilson coefficients,  $I$  is the unit operator,  $G_{\alpha\beta}$  is the gluon field strength tensor,  $\Gamma$  and  $\Gamma'$  are the matrices appearing in the calculations. Considering the vacuum expectation value of the OPE, the correlator function can be written in terms of the local operators as follows:

$$\Pi_{\nu\mu}(p_1^2, p_2^2, q^2) = (C_0)_{\nu\mu} + (C_3)_{\nu\mu} \langle \bar{q}q \rangle + (C_4)_{\nu\mu} \langle G^2 \rangle + (C_5)_{\nu\mu} \langle \bar{q} \sigma_{\alpha\beta} G^{\alpha\beta} q \rangle + (C_6)_{\nu\mu} \langle \bar{q} \Gamma q \bar{q} \Gamma' q \rangle. \quad (14)$$

The values of the heavy quark condensates are related to the vacuum expectation value of the gluon operators in the following manner:

$$\langle \bar{Q}Q \rangle = - \frac{1}{12m_Q} \frac{\alpha_s}{\pi} \langle G^2 \rangle - \frac{1}{360m_Q^3} \frac{\alpha_s}{\pi} \langle G^3 \rangle, \quad (15)$$

where  $Q$  is the heavy quark and the heavy quark condensate contributions are suppressed by inverse of the heavy quark mass, and for this reason we can safely omit them.

It should be noted here that, in principle, the light quark condensate does give a contribution to the correlator function, but its contribution becomes zero after a double Borel transformation. Therefore, the only non-perturbative contribution to the above-mentioned correlator function comes from the gluon condensate.

As a result, in the lowest order of perturbation theory, the three-point functions receive a contribution from the bare-loop and gluon condensates, i.e.,

$$\Pi_i(p_1^2, p_2^2, q^2) = \Pi_i^{\text{per}}(p_1^2, p_2^2, q^2) + \Pi_i^{(G^2)}(p_1^2, p_2^2, q^2) q \frac{\alpha_s}{\pi} \langle G^2 \rangle. \quad (16)$$

The bare-loop contribution can be obtained using the double dispersion representation

$$\Pi_i^{\text{per}} = -\frac{1}{(2\pi)^2} \int \frac{\rho_i^{\text{per}}(s, s', Q^2)}{(s-p^2)(s'-p'^2)} ds ds' + \text{subtraction terms}, \quad (17)$$

in the variable  $p^2$  and  $p'^2$ , where  $Q^2 = -q^2$ . The integration region in (17) is determined by the inequalities

$$-1 \leq \frac{2ss' + (s+s'+Q^2)(m_c^2 - m_b^2 - s) + 2sm_b^2}{\lambda^{1/2}(s, s', -Q^2)\lambda^{1/2}(m_b^2, m_c^2, s)} \leq 1, \quad (18)$$

where  $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ .

The spectral densities  $\rho_i^{\text{per}}(s, s', Q^2)$  can be calculated by using the Cutkovsky rule, i.e., by replacing the propagators with Dirac delta functions:

$$\frac{1}{k^2 - m^2} \rightarrow -2i\pi\delta(k^2 - m^2). \quad (19)$$

After standard calculations, we get for the spectral densities

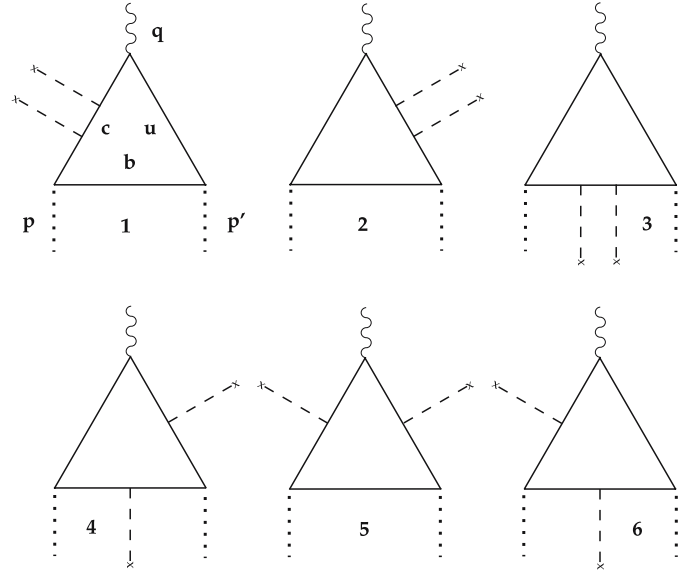
$$\rho_V = \frac{2N_c m_c}{\lambda^{3/2}(s, s', -Q^2)} (2s' \Delta_1 - u \Delta_2), \quad (20)$$

$$\begin{aligned} \rho_{A_1} = & -\frac{2N_c m_c}{\lambda^{3/2}(s, s', -Q^2)} \\ & \times \left[ m_c^4 s' + m_c^2 m_b^2 (u - 2s') + m_c^2 s' (u - 2s) \right. \\ & \left. + Q^2 (m_b^2 u - ss' - m_b^4) + \frac{1}{2} \lambda(s, s', -Q^2) \right], \quad (21) \end{aligned}$$

$$\rho_{A_2} = -\frac{N_c m_c}{\lambda^{3/2}(s, s', -Q^2)} \times \left[ -(2s \Delta_2 - u \Delta_1) + B_1 + C - D - E \right], \quad (22)$$

where  $N_c = 3$  is the color factor,  $u = s + s' + Q^2$ ,  $\Delta_1 = s + m_b^2 - m_c^2$ ,  $\Delta_2 = s' + m_b^2$ , and the explicit forms of the functions  $B$ ,  $C$ ,  $D$ , and  $E$  are given in Appendix B. According to the QCD sum rule philosophy, the contributions coming from the excited states are approximated as a bare-loop contribution, starting from some thresholds  $s$  and  $s'$ , in accordance with the quark-hadron duality. Note that we neglected  $\mathcal{O}(\alpha_s/\pi)$  hard gluon corrections to the bare-loop diagrams, since they are not available yet. However, we expect their contributions to be about  $\sim 10\%$ , so that if the accuracy of QCD sum rules is taken into account, these corrections would not change the results drastically.

The next problem is the calculation of the gluon condensate contributions to the correlation functions. The



**Fig. 1.** Gluon condensate contribution diagrams to the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay. In this figure the *dashed line* represents the soft gluon line,  $c, u, b$  identify the quark lines,  $p$  and  $p'$  are the four-momenta of the incoming  $B_c$  and outgoing  $B_u^*$  mesons, respectively, and  $q$  is the four-momentum of the electroweak current

gluon condensate contributions to the three-point sum rules are described by the diagrams presented in Fig. 1. The calculations of these diagrams are carried out in the Fock-Schwinger fixed-point gauge [20–22]

$$x^\mu A_\mu^a = 0,$$

where  $A_\mu^a$  is the gluon field. In calculating the gluon condensate contributions, integrals of the following types are encountered:

$$\begin{aligned} I_0[a, b, c] &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m_b^2]^a [(p+k)^2 - m_c^2]^b [(p'+k)^2]^c}, \\ I_\mu[a, b, c] &= \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu}{[k^2 - m_b^2]^a [(p+k)^2 - m_c^2]^b [(p'+k)^2]^c}, \\ I_{\mu\nu}[a, b, c] &= \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{[k^2 - m_b^2]^a [(p+k)^2 - m_c^2]^b [(p'+k)^2]^c}. \quad (23) \end{aligned}$$

These integrals can be calculated by continuing to Euclidean space-time and using the Schwinger representation for the Euclidean propagator:

$$\frac{1}{k^2 + m^2} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha(k^2 + m^2)}, \quad (24)$$

which is very suitable for the Borel transformation since

$$\mathcal{B}_{p^2}(M^2) e^{-\alpha p^2} = \delta(1/M^2 - \alpha). \quad (25)$$

Performing the integration over the loop momentum and over the two parameters which we have used in the exponential representation of the propagators [21], and applying double Borel transformations over  $p^2$  and  $p'^2$ , we get the Borel transformed form of the integrals in (23) (see also [21])

$$\begin{aligned}\hat{I}_0(a, b, c) &= \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{2-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-4, 1-c-b), \\ \hat{I}_\mu(a, b, c) &= \frac{1}{2} [\hat{I}_1(a, b, c) + \hat{I}_2(a, b, c)] \mathcal{P}_\mu \\ &\quad + \frac{1}{2} [\hat{I}_1(a, b, c) - \hat{I}_2(a, b, c)] q_\mu, \\ \hat{I}_{\mu\nu}(a, b, c) &= \hat{I}_6(a, b, c) g_{\mu\nu} + \frac{1}{4} (2\hat{I}_4 + \hat{I}_3 + \hat{I}_5) \mathcal{P}_\mu \mathcal{P}_\nu \\ &\quad + \frac{1}{4} (-\hat{I}_5 + \hat{I}_3) \mathcal{P}_\mu q_\nu + \frac{1}{4} (-\hat{I}_5 + \hat{I}_3) \mathcal{P}_\nu q_\mu \\ &\quad + \frac{1}{4} (-2\hat{I}_4 + \hat{I}_3 + \hat{I}_5) q_\mu q_\nu, \quad (26)\end{aligned}$$

where

$$\begin{aligned}\hat{I}_1(a, b, c) &= i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{3-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-5, 1-c-b), \\ \hat{I}_2(a, b, c) &= i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{2-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-5, 1-c-b), \\ \hat{I}_3(a, b, c) &= i \frac{(-1)^{a+b+c}}{32\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{4-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-6, 1-c-b), \\ \hat{I}_4(a, b, c) &= i \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{3-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-6, 1-c-b), \\ \hat{I}_5(a, b, c) &= i \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{4-a-b} (M_2^2)^{2-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-6, 1-c-b), \\ \hat{I}_6(a, b, c) &= i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{3-a-c} \\ &\quad \times \mathcal{U}_0(a+b+c-6, 2-c-b), \quad (27)\end{aligned}$$

where  $M_1^2$  and  $M_2^2$  are the Borel parameters in the  $s$  and  $s'$  channel, respectively, and the function  $\mathcal{U}_0(\alpha, \beta)$  is defined as

$$\begin{aligned}\mathcal{U}_0(a, b) &= \int_0^\infty dy (y + M_1^2 + M_2^2)^a y^b \\ &\quad \times \exp \left[ -\frac{B_{-1}}{y} - B_0 - B_1 y \right],\end{aligned}$$

where

$$B_{-1} = \frac{m_c^2}{M_1^2} [m_c^2 M_2^2 + M_1^2 (m_c^2 + Q^2)],$$

$$\begin{aligned}B_0 &= \frac{1}{M_1^2 M_2^2} [m_b^2 M_1^2 + M_2^2 (m_b^2 + m_c^2)], \\ B_1 &= \frac{m_b^2}{M_1^2 M_2^2}.\end{aligned} \quad (28)$$

The hat in (26) and (27) means that they are double Borel transformed form of integrals. Performing double Borel transformations on the variables  $p^2$  and  $p'^2$  on the physical parts of the correlator functions and bare-loop diagrams and equating two representations of the correlator functions, we get the sum rules for the form factors  $V$ ,  $A_1$  and  $A_2$ :

$$\begin{aligned}V &= -\frac{(m_b + m_c)(m_{B_c} + m_{B_u^*})}{2f_{B_c} m_{B_c}^2 f_{B_u^*} m_{B_u^*}} e^{m_{B_c}^2/M_1^2} e^{m_{B_u^*}^2/M_2^2} \\ &\quad \times \left\{ \frac{1}{4\pi^2} \int_{m_b^2}^{s'_0} ds' \int_{s_L}^{s_0} \rho_V(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} \right. \\ &\quad \left. - i \frac{1}{24\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle C_4^V \right\}, \\ A_1 &= \frac{(m_b + m_c)}{f_{B_c} m_{B_c}^2 f_{B_u^*} m_{B_u^*} (m_{B_c} + m_{B_u^*})} e^{m_{B_c}^2/M_1^2} e^{m_{B_u^*}^2/M_2^2} \\ &\quad \times \left\{ \frac{1}{4\pi^2} \int_{m_b^2}^{s'_0} ds' \int_{s_L}^{s_0} \rho_{A_1}(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} \right. \\ &\quad \left. - i \frac{1}{24\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle C_4^{A_1} \right\}, \\ A_2 &= -\frac{(m_b + m_c)(m_{B_c} + m_{B_u^*})}{f_{B_c} m_{B_c}^2 f_{B_u^*} m_{B_u^*} (m_{B_c}^2 + 3m_{B_u^*}^2 + Q^2)} \\ &\quad \times e^{m_{B_c}^2/M_1^2} e^{m_{B_u^*}^2/M_2^2} \\ &\quad \times \left\{ \frac{1}{4\pi^2} \int_{m_b^2}^{s'_0} ds' \int_{s_L}^{s_0} \rho_{A_2}(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} \right. \\ &\quad \left. - i \frac{1}{24\pi^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle C_4^{A_2} \right\},\end{aligned}$$

where  $s_0$  and  $s'_0$  are the continuum thresholds in the pseudoscalar  $B_c$  and  $B_u^*$  channels, respectively, and the lower bound integration limit of  $s$  is given as

$$s_L = \frac{(m_b^2 + q^2 - m_c^2 - s')(m_c^2 s' - m_b^2 q^2)}{(m_c^2 - q^2)(m_b^2 - s')}. \quad (29)$$

Explicit expressions of  $C_4^V$ ,  $C_4^{A_1}$  and  $C_4^{A_2}$  are all presented in Appendix C. At the end of this section we present the dilepton invariant mass distribution for the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay. Using the parametrization of the  $B_c \rightarrow B_u^*$  transition in terms of form factors and (1), the matrix element of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay can be written as

$$\mathcal{M} = \frac{G\alpha}{4\sqrt{2}\pi} m_{B_c} \left[ J_\mu^1 \bar{\ell} \gamma_\mu \ell + J_\mu^2 \bar{\ell} \gamma_\mu \gamma_5 \ell \right], \quad (30)$$

where

$$\begin{aligned} J_\mu^1 &= G_1(\hat{s}) \epsilon_{\mu\rho\alpha\beta} \epsilon^{*\rho} \hat{p}^\alpha \hat{p}'^\beta - iG_2(\hat{s}) \epsilon_\mu^* \\ &\quad + iG_3(\hat{s}) (\epsilon^* \hat{q}) \hat{P}_\mu + iG_4(\hat{s}) (\epsilon^* \hat{q}) \hat{q}_\mu, \\ J_\mu^2 &= H_1(\hat{s}) \epsilon_{\mu\rho\alpha\beta} \epsilon^{*\rho} \hat{p}^\alpha \hat{p}'^\beta - iH_2(\hat{s}) \epsilon_\mu^* \\ &\quad + iH_3(\hat{s}) (\epsilon^* \hat{q}) \hat{P}_\mu + iH_4(\hat{s}) (\epsilon^* \hat{q}) \hat{q}_\mu, \end{aligned} \quad (31)$$

where  $\hat{P}_\mu = \mathcal{P}_\mu/m_{B_c}$ ,  $\hat{q}_\mu = q_\mu/m_{B_c}$  and  $\hat{s} = q^2/m_{B_c}^2$ , and

$$\begin{aligned} G_1(\hat{s}) &= \frac{2}{1+\hat{r}} C_9^{\text{eff}} V(\hat{s}) + \frac{4\hat{r}_c}{\hat{s}} C_7^{\text{eff}} T_1(\hat{s}), \\ G_2(\hat{s}) &= (1+\hat{r}) \left[ C_9^{\text{eff}} A_1(\hat{s}) + \frac{2\hat{r}_c}{\hat{s}} (1-\hat{r}) C_7^{\text{eff}} T_2(\hat{s}) \right], \\ G_3(\hat{s}) &= \frac{1}{1-\hat{r}^2} \left\{ (1-\hat{r}) C_9^{\text{eff}} A_2(\hat{s}) \right. \\ &\quad \left. + 2\hat{r}_c C_7^{\text{eff}} \left[ T_3(\hat{s}) + \frac{1-\hat{r}^2}{\hat{s}} T_2(\hat{s}) \right] \right\}, \\ G_4(\hat{s}) &= \frac{1}{\hat{s}} \left\{ C_9^{\text{eff}} \left[ (1+\hat{r}) A_1(\hat{s}) - (1-\hat{r}) A_2(\hat{s}) - 2\hat{r} A_0(\hat{s}) \right] \right. \\ &\quad \left. - 2\hat{r}_c C_7^{\text{eff}} T_3(\hat{s}) \right\}, \\ H_1(\hat{s}) &= \frac{2}{1+\hat{r}} C_{10} V(\hat{s}), \\ H_2(\hat{s}) &= (1+\hat{r}) C_{10} A_1(\hat{s}), \\ H_3(\hat{s}) &= \frac{1}{1+\hat{r}} C_{10} A_2(\hat{s}), \\ H_4(\hat{s}) &= \frac{1}{\hat{s}} C_{10} \left[ (1+\hat{r}) A_1(\hat{s}) - (1-\hat{r}) A_2(\hat{s}) - 2\hat{r} A_0(\hat{s}) \right], \end{aligned} \quad (32)$$

where  $\hat{r} = m_{B_u^*}/m_{B_c}$  and  $\hat{r}_c = m_c/m_{B_c}$ .

Using (30), the dilepton invariant mass distribution takes the following form:

$$\begin{aligned} \frac{d\Gamma}{d\hat{s}} &= \frac{G^2 \alpha_s^2}{2^{10} \pi^5} m_{B_c}^5 \sqrt{\lambda} v \left\{ \frac{1}{3} \hat{s} \lambda \left( 1 + 2 \frac{\hat{r}_\ell^2}{\hat{s}} \right) |G_1|^2 + \frac{1}{3} \hat{s} \lambda v^2 |H_1|^2 \right. \\ &\quad + \frac{1}{4\hat{r}^2} \left[ \left( \lambda - \frac{\lambda v^2}{3} + 8\hat{r}^2(\hat{s} + 2\hat{r}_\ell^2) \right) |G_2|^2 \right. \\ &\quad \left. + \left( \frac{\lambda}{3}(3-v^2) + 8\hat{r}^2 \hat{s} v^2 \right) |H_2|^2 \right] \\ &\quad + \frac{\lambda}{4\hat{r}^2} \left[ \frac{\lambda}{3}(3-v^2) |G_3|^2 \right. \\ &\quad \left. + \left( \frac{\lambda}{3}(3-v^2) + 4\hat{r}_\ell^2(2+2\hat{r}^2-\hat{s}) \right) |H_3|^2 \right] \\ &\quad - \frac{1}{2\hat{r}^2} \left[ \frac{\lambda}{3}(3-v^2)(1-\hat{r}^2-\hat{s}) \text{Re}[G_2 G_3^*] \right. \\ &\quad \left. + \left( \frac{\lambda}{3}(3-v^2)(1-\hat{r}^2-\hat{s}) + 4\hat{r}_\ell^2 \lambda \right) \text{Re}[H_2 H_3^*] \right] \end{aligned}$$

$$\begin{aligned} &- 2\lambda \frac{\hat{r}_\ell^2}{\hat{r}^2} \left( \text{Re}[H_2 H_4^*] - (1-\hat{r}^2) \text{Re}[H_3 H_4^*] \right) \\ &\quad + \frac{\hat{r}_\ell^2}{\hat{r}^2} \hat{s} \lambda |H_4|^2 \Big\}, \end{aligned} \quad (33)$$

where

$$\lambda = \lambda(1, \hat{r}^2, \hat{s}),$$

and

$$v = \sqrt{1 - \frac{4m_\ell^2}{q^2}}$$

is the lepton velocity, and finally

$$r_\ell^2 = \frac{m_\ell^2}{m_{B_c}^2}.$$

### 3 Numerical analysis

In this section we present our numerical calculation of the form factors  $A_1$ ,  $A_2$ ,  $A_0$ ,  $V$ ,  $T_1$ ,  $T_2$  and  $T_3$ . The values of the input parameters appearing in the sum rules for the form factors are  $m_{B_c} = 6.4$  GeV,  $m_{B_u^*} = 5.325$  GeV,  $f_{B_c} = 0.385$  GeV [24],  $f_{B_u^*} = 160$  MeV [25],  $m_b = 4.8$  GeV,  $m_c(\mu = m_c) = 1.26$  GeV,  $\langle(\alpha_s/\pi)G^2\rangle = 0.012$  GeV<sup>4</sup>[19]. The decay constants of the  $B_c$  and  $B_u^*$  mesons are determined from the two-point QCD sum rules. As has already been noted, in bare-loop calculations we neglect  $\mathcal{O}(\alpha_s/\pi)$  corrections. For consistency, these corrections are also neglected in the calculations for the leptonic decay constants.

The parameters  $s_0$  and  $s'_0$ , which are the continuum thresholds of  $B_c$  and  $B_u^*$  mesons, respectively, are also determined from the two-point QCD sum rules, and they are taken to be  $s_0 = 50$  GeV<sup>2</sup> and  $s'_0 = 35$  GeV<sup>2</sup>. These continuum thresholds are determined from the conditions that guarantees the sum rules to have the best stability in the allowed  $M_1^2$  and  $M_2^2$  region.

The Borel parameters  $M_1^2$  and  $M_2^2$  in the sum rules are auxiliary parameters, and physical quantities should be independent of them. Therefore it is necessary to look for working regions of these parameters where physical results exhibit the best stability. The working regions of  $M_1^2$  and  $M_2^2$  are determined by requiring that the continuum and higher states contributions are effectively suppressed, which ensures that the results do not sensitively depend on such excited states. We require also that the contribution gluon condensate is not too large, which guarantees that the contributions of higher dimensional operators are small. Our analysis verifies that the working regions  $20 \text{ GeV}^2 \leq M_1^2 \leq 40 \text{ GeV}^2$  and  $10 \text{ GeV}^2 \leq M_2^2 \leq 15 \text{ GeV}^2$  of the Borel parameters satisfy both of the above-mentioned requirements.

In these regions of  $M_1^2$  and  $M_2^2$ , the gluon condensate contribution constitutes approximately 5%, and higher state contributions constitute at most 30% of the total result.

Our numerical results for the form factors at  $q^2 = 0$  are

$$\begin{aligned} V(0) &= 0.09 \pm 0.01, \\ A_1(0) &= -0.17 \pm 0.02, \end{aligned}$$

$$\begin{aligned} A_2(0) &= 1.10 \pm 0.10, \\ T_1(0) = T_2(0) = -A_0(0) &= 0.23 \pm 0.03, \end{aligned} \quad (34)$$

and the value of the form factor  $T_3(0)$  can easily be obtained from (7).

The errors are estimated by the variation of the Borel parameters  $M_1^2$  and  $M_2^2$ , the variation of the continuum thresholds  $s_0$  and  $s'_0$ , the variation of the  $b$  and  $c$  quark masses and the leptonic decay constants  $f_{B_c}$  and  $f_{B_u^*}$ . The main uncertainty comes from the thresholds and the decay constants, which is about  $\sim 20\%$  of the central value, while the other uncertainties are small, constituting a few percent. Note that all the uncertainties are added quadratically. Here we would like to make the following cautionary note. It is well known that for heavy quarkonia, where the quark velocities are small, the  $\alpha_s/v$  corrections caused by the Coulomb-like interaction of quarks are essential, where  $v$  is the quark velocity. In our case, we have two expansion parameters,  $\alpha_s/v_1$  and  $\alpha_s/v_2$ , where  $v_1$  and  $v_2$  are the relative velocities of the quarks ( $b\bar{c}$ ) and ( $b\bar{u}$ ) (for the massless  $u$  quark  $v_2 = 1$ ). When these corrections are taken into account the value of the form factors at  $Q^2 = 0$  are twice as large. In the further numerical analysis we will omit the dependence of the form factors on  $q^2$ , which gives small contributions to the overall result. Indeed, the maximum value of  $q^2$  in the decay under consideration is about  $(m_{B_c} - m_{B_u^*})^2 \sim 1 \text{ GeV}^2$  and assuming that the simple pole model correctly describes the  $q^2$  dependence of the form factors, it is easy to see that  $q^2/m_{B_c}^2$  is about  $1/36$ ; i.e., the results could be changed maximally by about  $3\%$ .

Integrating (32) over  $q^2$  in the whole physical region and using the total mean life time  $\tau \simeq 0.46 \text{ ps}$  of the  $B_c$  meson [26], the branching ratio of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay is

$$\mathcal{B}(B_c \rightarrow B_u^* \ell^+ \ell^-) = \begin{cases} 1.30 \times 10^{-9}, \\ 3.88 \times 10^{-7}, \end{cases}$$

where the upper value corresponds to the case when only short distance contributions are taken into account, and the lower one corresponds to the case when short and long distance contributions due to the  $\rho$  and  $\omega$  resonances are taken into account.

It follows from this result that the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay with the above-presented width can be measurable at LHC.

In conclusion, we calculated the  $B_c \rightarrow B_u^*$  transition form factors  $V(q^2)$ ,  $A_1(q^2)$ ,  $A_2(q^2)$  and  $T_1(q^2)$  in the framework of QCD sum rules. Our calculations show that this rare semileptonic decay can be measurable at LHC. Using the results of the form factors calculated in the present work, we estimate the branching ratio of the  $B_c \rightarrow B_u^* \ell^+ \ell^-$  decay.

*Acknowledgements.* One of the authors (T.M.A.) is grateful to S. Fajfer for useful discussion and TÜBİTAK for partial support of this work under project 105T131.

## Appendix A

In this appendix we present the expressions of the functions  $F_1$ ,  $F_2$ ,  $C^{\text{box}}$  and  $C^Z$ , which enter the expressions of

the Wilson coefficients  $C_7(m_W)$ ,  $C_9(m_W)$  and  $C_{10}(m_W)$ :

$$\begin{aligned} C^{\text{box}}(x_i) &= \frac{3}{8} \left[ -\frac{1}{x_i - 1} + \frac{x_i \ln x_i}{(x_i - 1)^2} \right], \\ C^Z(x_i) &= \frac{x_i}{4} - \frac{3}{8} \frac{1}{x_i - 1} + \frac{3}{8} \frac{2x_i^2 - x_i}{8(x_i - 1)^2} \ln x_i, \\ F_1(x_i) &= Q_d \left\{ \left[ \frac{1}{12} \frac{1}{x_i - 1} + \frac{13}{12} \frac{1}{(x_i - 1)^2} - \frac{1}{2(x_i - 1)^3} \right] x_i \right. \\ &\quad + \left[ \frac{2}{3} \frac{1}{x_i - 1} + \left( \frac{2}{3} \frac{1}{(x_i - 1)^2} - \frac{5}{6} \frac{1}{(x_i - 1)^3} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{1}{(x_i - 1)^4} \right) x_i \right] \ln x_i \left. \right\} \\ &\quad - \left[ \frac{7}{3} \frac{1}{x_i - 1} + \frac{13}{12} \frac{1}{(x_i - 1)^2} - \frac{1}{2} \frac{1}{(x_i - 1)^3} \right] x_i \\ &\quad - \left[ \frac{1}{6} \frac{1}{x_i - 1} - \frac{35}{12} \frac{1}{(x_i - 1)^2} - \frac{5}{6} \frac{1}{(x_i - 1)^3} \right. \\ &\quad \left. + \frac{1}{2} \frac{1}{(x_i - 1)^4} \right] x_i \ln x_i, \\ F_2(x_i) &= -Q_d \left\{ \left[ -\frac{1}{4} \frac{1}{x_i - 1} + \frac{3}{4} \frac{1}{(x_i - 1)^2} + \frac{3}{2} \frac{1}{(x_i - 1)^3} \right] \right. \\ &\quad \left. - \frac{3}{2} \frac{x_i^2 \ln x_i}{(x_i - 1)^4} \right\} \\ &\quad + \left[ \frac{1}{2} \frac{1}{x_i - 1} + \frac{9}{4} \frac{1}{(x_i - 1)^2} + \frac{3}{2} \frac{1}{(x_i - 1)^3} \right] x_i \\ &\quad - \frac{3}{2} \frac{x_i^3 \ln x_i}{2(x_i - 1)^4}, \end{aligned} \quad (\text{A.1})$$

where  $x_i = m_i^2/m_W^2$ , and  $Q_d$  is the down quark charge. Note that, in these expressions, we omit the gauge dependent terms  $\gamma(\xi, x_i)$  [16], because these terms are canceled out in the combinations  $C^{\text{box}}(x_i) + C^Z(x_i)$  and  $F_1(x_i) + C^Z(x_i)$ .

## Appendix B

In this appendix we present the expressions of the functions  $B_1$ ,  $C$ ,  $D$ , and  $E$  which appear in the calculations of the spectral density  $\rho_{A_2}$  (see (22)). We have

$$\begin{aligned} B_1 &= \frac{1}{\lambda(s, s', -Q^2)} I_0 \left\{ m_b^2 \left[ (Q^2 + s)^2 - 2s'(2Q^2 + s) + s'^2 \right] \right. \\ &\quad + 2m_b^2 s' \left[ Q^4 - (s - s')^2 + 3m_c^2 (u - 2s') + Q^2 u \right] \\ &\quad + s'^2 \left[ 6m_c^4 + Q^4 + (s - s')^2 + 6m_c^2 (u - 2s) \right. \\ &\quad \left. \left. + 2Q^2 (s - 2s') \right] \right\}, \\ C = D &= \frac{1}{\lambda^2(s, s', -Q^2)} I_0 \\ &\quad \times \left\{ m_b^2 (u - s') \left[ -Q^4 + m_b^2 (2Q^2 - s) + s^2 \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + s' \left[ s(2Q^2 - s)(Q^2 + s) + m_b^4(Q^2 + 2s) \right. \\
& \left. - m_b^2(Q^4 + 6Q^2s + s^2) \right] \\
& + s'^2 \left[ m_b^4 + m_b^2(s - Q^2) - s(Q^2 + 2s) \right] \\
& + s'^3 (m_b^2 - s) - 3m_c^4 s' u \\
& + 2m_c^2 \left[ -m_b^2 \left( (u - s')^2 + s'(s - Q^2) - 2s' \right) \right. \\
& \left. - s' \left( Q^2(u - 2s) - 2s^2 + s'u \right) \right] \Big\}, \\
E = & \frac{1}{\lambda^{3/2}(s, s', -Q^2)} I_0 \left\{ s^2 \left[ m_c^4(u + 6ss') \right. \right. \\
& + 2m_b^2 s \left[ Q^4 - (s - s')^2 + Q^2 u \right] \\
& + m_b^4 \left[ Q^4 + (s - s')^2 + 2Q^2(-2s + s') \right] \\
& + 2m_c^2 \left[ -m_b^3 \left( q^2(u - 2s) - 2s^2 + s'u \right) \right. \\
& \left. + s \left( -(u - s')^2 + s'(-s + Q^2) + 2s'^2 \right) \right] \Big\}, \quad (\text{B.1})
\end{aligned}$$

where

$$I_0 = \frac{1}{4\lambda^{1/2}(s, s', -Q^2)}.$$

## Appendix C

In this appendix we give the explicit expressions of the coefficients of the gluon condensate which enter the sum rules for the form factors  $V$ ,  $A_1$  and  $A_2$ , respectively:

$$\begin{aligned}
C_4^V = & 192m_c \hat{I}_1(1, 3, 1) + 192m_c^3 \hat{I}_1(1, 4, 1) \\
& - 32m_c \hat{I}_1(2, 1, 2) + 64m_c \hat{I}_1(1, 1, 3) \\
& + 32m_c \hat{I}_1(2, 1, 2) + 128m_c m_b^2 \hat{I}_1(2, 1, 3) \\
& - 128m_c \hat{I}_1^{[0,1]}(2, 1, 3) + 64m_c \hat{I}_1(3, 1, 1) \\
& + 192m_c m_b^2 \hat{I}_1(3, 1, 2) - 128m_c \hat{I}_1^{[0,1]}(3, 1, 2) \\
& + 64m_c m_b^4 \hat{I}_1(3, 1, 3) - 128m_c m_b^2 \hat{I}_1^{[0,1]}(3, 1, 3) \\
& + 64m_c \hat{I}_1^{[0,2]}(3, 1, 3) + 32m_c \hat{I}_1(1, 2, 2) \\
& + 32m_c \hat{I}_1(2, 2, 1) + 32m_c m_b^2 \hat{I}_1(2, 2, 2) \\
& - 32m_c \hat{I}_1^{[0,1]}(2, 2, 2) - 32m_c \hat{I}_1(1, 2, 2) \\
& + 192m_c m_b^2 \hat{I}_1(4, 1, 1) + 96m_c \hat{I}_1(2, 2, 1), \quad (\text{C.1})
\end{aligned}$$

$$\begin{aligned}
C_4^{A_1} = & 96m_c m_b^2 \hat{I}_0(1, 3, 1) - 96m_c \hat{I}_0^{[0,1]}(1, 3, 1) \\
& + 96m_c^3 m_b^2 \hat{I}_0(1, 4, 1) - 96m_c^3 \hat{I}_0^{[0,1]}(1, 4, 1) \\
& - 384m_c \hat{I}_6(1, 3, 1) - 384m_c^3 \hat{I}_6(1, 4, 1) \\
& - 16m_c \hat{I}_0(1, 1, 2) - 16m_c \hat{I}_0(2, 1, 1) \\
& - 16m_c m_b^2 \hat{I}_0(2, 1, 2) + 16m_c \hat{I}_0^{[0,1]}(2, 1, 2) \\
& - 64m_c \hat{I}_6(2, 1, 2) - 16m_c \hat{I}_0(1, 1, 2) \\
& + 96m_c m_b^2 \hat{I}_0(1, 1, 3) - 96m_c \hat{I}_0^{[0,1]}(1, 1, 3) \\
& - 16m_c \hat{I}_0(2, 1, 1) + 48m_c m_b^2 \hat{I}_0(2, 1, 2)
\end{aligned}$$

$$\begin{aligned}
& - 80m_c \hat{I}_0^{[0,1]}(2, 1, 2) + 96m_c m_b^4 \hat{I}_0(2, 1, 3) \\
& - 192m_c m_b^2 \hat{I}_0^{[0,1]}(2, 1, 3) + 96m_c \hat{I}_0^{[0,2]}(2, 1, 3) \\
& + 64m_c m_b^2 \hat{I}_0(3, 1, 1) - 96m_c \hat{I}_0^{[0,1]}(3, 1, 1) \\
& + 64m_c m_b^4 \hat{I}_0(3, 1, 2) - 160m_c m_b^2 \hat{I}_0^{[0,1]}(3, 1, 2) \\
& + 96m_c \hat{I}_0^{[0,2]}(3, 1, 2) + 32m_c m_b^6 \hat{I}_0(3, 1, 3) \\
& - 96m_c m_b^4 \hat{I}_0^{[0,1]}(3, 1, 3) + 96m_c m_b^2 \hat{I}_0^{[0,2]}(3, 1, 3) \\
& - 32m_c \hat{I}_0^{[0,3]}(3, 1, 3) + 64m_c \hat{I}_6(2, 1, 2) \\
& - 128m_c m_b^2 \hat{I}_6(2, 1, 3) + 128m_c \hat{I}_6^{[0,1]}(2, 1, 3) \\
& - 256m_c m_b^2 \hat{I}_6(3, 1, 2) + 128m_c \hat{I}_6^{[0,1]}(3, 1, 2) \\
& - 128m_c m_b^4 \hat{I}_6(3, 1, 3) + 256m_c m_b^2 \hat{I}_6^{[0,1]}(3, 1, 3) \\
& - 128m_c \hat{I}_6^{[0,1]}(3, 1, 3) + 32m_c m_b^2 \hat{I}_0(1, 2, 2) \\
& - 32m_c \hat{I}_0^{[0,1]}(1, 2, 2) - 32m_c \hat{I}_0^{[0,1]}(2, 2, 1) \\
& + 16m_c m_b^4 \hat{I}_0(2, 2, 2) - 32m_c m_b^2 \hat{I}_0^{[0,1]}(2, 2, 2) \\
& + 16m_c \hat{I}_0^{[0,2]}(2, 2, 2) - 128m_c \hat{I}_6(1, 2, 2) \\
& + 64m_c m_b^2 \hat{I}_6(2, 2, 2) + 64m_c \hat{I}_6^{[0,1]}(2, 2, 2) \\
& - 16m_c \hat{I}_0(1, 2, 1) - 16m_c m_b^2 \hat{I}_0(1, 2, 2) \\
& + 16m_c \hat{I}_0^{[0,1]}(1, 2, 2) - 64m_c \hat{I}_6(1, 2, 2) \\
& + 96m_c m_b^2 \hat{I}_0(3, 1, 1) + 96m_c m_b^4 \hat{I}_0(4, 1, 1) \\
& - 96m_c m_b^2 \hat{I}_0^{[0,1]}(4, 1, 1) - 384m_c m_b^2 \hat{I}_6(4, 1, 1) \\
& + 48m_c \hat{I}_0(1, 2, 1) + 48m_c m_b^2 \hat{I}_0(2, 2, 1) \\
& - 48m_c \hat{I}_0^{[0,1]}(2, 2, 1) - 192m_c \hat{I}_6(2, 2, 1), \quad (\text{C.2})
\end{aligned}$$

$$\begin{aligned}
C_4^{A_2} = & 96m_c \hat{I}_2(1, 3, 1) + 96m_c^3 \hat{I}_2(1, 4, 1) \\
& - 96m_c \hat{I}_3(1, 3, 1) - 96m_c^3 \hat{I}_3(1, 4, 1) \\
& + 96m_c \hat{I}_5(1, 3, 1) + 96m_c^3 \hat{I}_5(1, 4, 1) \\
& + 16m_c \hat{I}_2(2, 1, 2) - 16m_c \hat{I}_3(2, 1, 2) \\
& + 16m_c \hat{I}_5(2, 1, 2) - 32m_c \hat{I}_0(1, 1, 3) \\
& - 32m_c \hat{I}_0(2, 1, 2) - 32m_c m_b^2 \hat{I}_0(2, 1, 3) \\
& + 32m_c \hat{I}_0^{[0,1]}(2, 1, 3) - 32m_c \hat{I}_2(1, 1, 3) \\
& - 16m_c \hat{I}_2(2, 1, 2) + 32m_c \hat{I}_2(3, 1, 1) \\
& + 96m_c m_b^2 \hat{I}_2(3, 1, 2) - 64m_c \hat{I}_2^{[0,1]}(3, 1, 2) \\
& + 32m_c m_b^4 \hat{I}_2(3, 1, 3) - 64m_c m_b^2 \hat{I}_2^{[0,1]}(3, 1, 3) \\
& + 32m_c \hat{I}_2^{[0,2]}(3, 1, 3) + 16m_c \hat{I}_2(1, 2, 2) \\
& - 32m_c m_b^2 \hat{I}_3(2, 1, 3) + 32m_c \hat{I}_3^{[0,1]}(2, 1, 3) \\
& - 64m_c m_b^2 \hat{I}_3(3, 1, 2) + 32m_c \hat{I}_3^{[0,1]}(3, 1, 2) \\
& - 32m_c m_b^4 \hat{I}_3(3, 1, 3) + 64m_c m_b^2 \hat{I}_3^{[0,1]}(3, 1, 3) \\
& - 32m_c \hat{I}_3^{[0,2]}(3, 1, 3) - 16m_c \hat{I}_5(2, 1, 2) \\
& + 32m_c m_b^2 \hat{I}_5(2, 1, 3) - 32m_c \hat{I}_5^{[0,1]}(2, 1, 3) \\
& + 64m_c m_b^2 \hat{I}_5(3, 1, 2) - 32m_c \hat{I}_5^{[0,1]}(3, 1, 2) \\
& + 32m_c m_b^4 \hat{I}_5(3, 1, 3) - 64m_c m_b^2 \hat{I}_5^{[0,1]}(3, 1, 3) \\
& + 32m_c \hat{I}_5^{[0,2]}(3, 1, 3) + 16m_c \hat{I}_0(1, 2, 2) \\
& - 32m_c m_b^2 \hat{I}_0(2, 2, 2) + 48m_c \hat{I}_2(1, 2, 2)
\end{aligned}$$



$$\begin{aligned}
& + 16m_c \hat{I}_2(2, 2, 1) - 48m_c m_b^2 \hat{I}_2(2, 2, 2) \\
& - 16m_c \hat{I}_2^{[0,1]}(2, 2, 2) - 32m_c \hat{I}_3(1, 2, 2) \\
& + 16m_c m_b^2 \hat{I}_3(2, 2, 2) + 16m_c \hat{I}_3^{[0,1]}(2, 2, 2) \\
& + 32m_c \hat{I}_5(1, 2, 2) - 16m_c m_b^2 \hat{I}_5(2, 2, 2) \\
& - 16m_c \hat{I}_5^{[0,1]}(2, 2, 2) - 32m_c \hat{I}_1(1, 2, 2) \\
& + 16m_c \hat{I}_2(1, 2, 2) - 16m_c \hat{I}_3(1, 2, 2) \\
& + 16m_c \hat{I}_5(1, 2, 2) + 96m_c m_b^2 \hat{I}_2(4, 1, 1) \\
& - 96m_c m_b^2 \hat{I}_3(4, 1, 1) + 96m_c m_b^2 \hat{I}_5(4, 1, 1) \\
& + 48m_c \hat{I}_2(2, 2, 1) - 48m_c \hat{I}_3(2, 2, 1) \\
& + 48m_c \hat{I}_5(2, 2, 1), \tag{C.3}
\end{aligned}$$

where

$$\begin{aligned}
\hat{I}_n^{[i,j]}(a, b, c) &= (M_1^2)^i (M_2^2)^j \frac{d^i}{d(M_1^2)^i} \frac{d^j}{d(M_2^2)^j} \\
&\times \left[ (M_1^2)^i (M_2^2)^j \hat{I}_n(a, b, c) \right].
\end{aligned}$$

## References

1. CDF Collaboration, F. Abe et al., Phys. Rev. D **58**, 112004 (1998); Phys. Rev. Lett. **81**, 2432 (1998)
2. CDF Collaboration, A. Abulencia et al., hep-ex/0505076 (2005)
3. V.V. Kiselev, hep-ph/0308214 (2003)
4. I.P. Gouzev, V.V. Kiselev, A.K. Likhoded, V.I. Romanovsky, O.P. Yushchenko, Phys. Atom. Nucl. **67**, 1559 (2004)
5. Y. Grossman, Int. J. Mod. Phys. A **19**, 907 (2004)
6. G. Isidori, hep-ph/0401079 (2004)
7. A.J. Buras, hep-ph/0402191 (2004)
8. T. Ohl et al., Nucl. Phys. B **403**, 605 (1993); J. Donoghue, E. Golowich, B. Holstein, J. Trampetic, Phys. Rev. D **33**, 179 (1986); A.A. Petrov, Phys. Rev. D **56**, 1685 (1997)
9. E. Golowich, A.A. Petrov, Phys. Lett. B **427**, 172 (1998); A. Falk, Y. Grossman, Z. Ligeti, A.A. Petrov, Phys. Rev. D **69**, 114021 (2004)
10. G. Burdman, E. Golowich, J. Hewett, S. Pakvasa, Phys. Rev. D **52**, 6383 (1995)
11. C. Greub et al., Phys. Lett. B **382**, 415 (1996)
12. B. Bajc, S. Fajfer, R.J. Oakes, Phys. Rev. D **54**, 5883 (1996); S. Fajfer, S. Prelovsek, P. Singer, Eur. Phys. J. C **6**, 471 (1999)
13. S. Fajfer, S. Prelovsek, P. Singer, Phys. Rev. D **59**, 114003 (1999)
14. T.M. Aliev, M. Savcı, Phys. Lett. B **480**, 97 (2000)
15. G. Burdman, E. Golowich, J. Hewett, S. Pakvasa, Phys. Rev. D **66**, 014009 (2002)
16. T. Inami, C.S. Lim, Prog. Theor. Phys. **65**, 197 (1981); Errata **65**, 1172 (1981)
17. C.S. Lim, T. Morozumi, A.I. Sanda, Phys. Lett. B **218**, 343 (1989); N.G. Deshpande, J. Trampetic, K. Panrose, Phys. Rev. D **39**, 1461 (1989)
18. M. Soares, Phys. Rev. D **54**, 6837 (1996)
19. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147**, 385 (1979)
20. V.A. Fock, Sov. Phys. **12**, 404 (1937)
21. J. Schwinger, Phys. Rev. **82**, 664 (1951)
22. V. Smilga, Sov. J. Nucl. Phys. **35**, 215 (1982)
23. V.V. Kiselev, A.K. Likhoded, A.I. Onishchenko, Nucl. Phys. B **569**, 473 (2000)
24. V.V. Kiselev, A.V. Tkabladze, Sov. J. Nucl. Phys. **50**, 1063 (1989); T.M. Aliev, O. Yilmaz, Nuove Cimento A **105**, 827 (1992)
25. V.M. Belyaev, V.M. Braun, A. Khodjamirian, R. Rückl, Phys. Rev. D **51**, 6177 (1995)
26. CDF Collaboration, A. Abulencia, hep-ex/0603027 (2006)